FLOW WITH A DETACHED SHOCK WAVE ABOUT A SYMMETRICAL PROFILE

(OBTEKANIE SIMMETRICHNOGO PROFILIA S OTOSHEDSHEI UDARNOI VOLNOI)

PMM Vol.22, No.2, 1958, pp.206-219

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(Received 2 July 1958)

The problem of flow with a detached shock wave past a body has been examined by many authors, but in an approximate formulation in most cases.

In [1,2], and in other publications the solution is given in the form of series. However, this method enables one to investigate only a small portion of the flow in the neighborhood of the axis of symmetry, and near the stagnation point. Making use of the hodograph plane for rotational flows, it is possible to examine only cases where the Mach number of the incident flow is close to one (Busemann [3] and others). The Japanese scientists, Tamada [4] and Hida [5], solve the given problem by making *a priori* assumptions about the form of the shock wave, and assuming the fluid to be incompressible behind the shock wave. Uchida and Yasuhara [6] propose a semi-graphical approximate method for computing the flow behind a curved shock wave, and Mitchell [1] calculates the flow around a blunt body by a difference method, using experimental data for the form and the position of the shock wave.

Below, the problem is solved numerically with the aid of integral relations proposed by Dorodnitsin [8]. This method reduces the problem of integrating a system of nonlinear partial differential equations to that of numerically solving an approximating system of ordinary differential equations. With the use of electronic computing machines, the method of integral relations affords the possibility of obtaining the final results with the required degree of accuracy for the problem in the exact formulation.

1. Formulation of the problem. Let us examine the flow with a detached shock wave past a plane body of arbitrary shape (profile), having an axis of symmetry. Suppose a uniform supersonic stream ($M_\infty > 1$) of an ideal gas, with constant velocity w_∞, flows past such a body at a zero angle of attack. A shock wave, the position and shape of which are not known in advance, is formed in front of the body. It is required to

Reprint Order No. PMM 23.

compute a mixed rotational flow of a compressible gas in the minimal domain of influence, i.e. in the domain bounded by the shock wave, the axis of symmetry, the body contour, and the first limiting characteristic (or characteristics) passing between the shock wave and the body.

Let us introduce dimensionless quantities, referring the speed to the maximum speed, the pressure and the density to the pressure and the density in front of the shock wave, and the linear dimensions to a typical body dimension; then we have the equations of motion, the equation of continuity, and the adiabatic equation in the form

$$\operatorname{rot}\,\mathbf{w}\times\mathbf{w}+\frac{\nabla w^2}{2}+\frac{\nabla(kp)}{\rho}=0\qquad\left(k=\frac{\varkappa-1}{2\varkappa}\right)$$

$$\nabla(\rho\mathbf{w})=0,\qquad \mathbf{w}\nabla\frac{p}{\rho^{\varkappa}}=0\tag{1.1}$$

where w, p, and ρ are the velocity, pressure, and density behind the shock front, and κ is the adiabatic exponent (for air $\kappa = 1.40$).

Let us write the system of equations (1.1) in polar coordinates r, θ (Fig.1), introducing the Bernoulli equation in place of one of the equations of motion; together with the equation for the stream function, ψ, the new system will have the form:

$$\frac{\partial rH}{\partial r}+\frac{\partial S}{\partial\vartheta}=g,\qquad \frac{\partial rh}{\partial r}+\frac{\partial t}{\partial\vartheta}=0$$

$$\frac{d\psi}{d\vartheta}=\rho\left(v\frac{dr}{d\vartheta}-ru\right),\qquad \varphi\equiv\frac{p}{\rho^{\varkappa}}=\varphi(\psi)\tag{1.2}$$

Here

$$H=kp+\rho u^2,\qquad S=\rho uv,\qquad g=kp+\rho v^2,\qquad h=\tau u,\qquad t=\tau v$$

$$p=(1-w^2)\rho,\qquad \rho=\tau\varphi^{-\frac{1}{\varkappa-1}},\qquad \tau=(1-w^2)^{\frac{1}{\varkappa-1}}$$

and u, v, are the components of the velocity, w, in the r and θ directions. In this system the unknown functions are u, v, ψ, ϕ.

The boundary conditions on the body contour $r = r(\theta)$ have the form

$$u=\frac{v}{r_0}\frac{dr_0}{d\vartheta},\qquad \psi=0$$

$$\varphi=\frac{4\varkappa}{\varkappa^2-1}\left(\frac{\varkappa-1}{\varkappa+1}\right)^{\varkappa}\frac{1}{w_\infty^{2\varkappa}}\left[\frac{w_\infty^2}{1-w_\infty^2}-\frac{(\varkappa-1)^2}{4\varkappa}\right]=\text{const}\tag{1.3}$$

Let the equation of the shock front be represented in the form $r = r_0(\theta) + \epsilon(\theta)$, where $\epsilon(\theta)$ is the distance from the body contour to the shock wave along the ray θ = const. We can express the well known shock conditions in the required form:

$$w_x = w_\infty\left[1 - \frac{2\sin^2\sigma}{\varkappa+1}\left(1 - \frac{1}{M_\infty^2 \sin^2\sigma}\right)\right] \qquad (1.4)$$

$$w_y = \frac{w_\infty}{\varkappa+1}\sin 2\sigma\left(1 - \frac{1}{M_\infty^2\sin^2\sigma}\right) \qquad (1.5)$$

$$\varphi = \frac{4\varkappa}{\varkappa^2-1}\left(\frac{\varkappa-1}{\varkappa+1}\right)^\varkappa\left[\omega - \frac{(\varkappa-1)^2}{4\varkappa}\right]\left(1+\frac{1}{\omega}\right)^\varkappa \qquad (1.6)$$

$$\psi \equiv \psi_\infty = w_\infty(1 - w_\infty^2)^{\frac{1}{\varkappa-1}}(r_0+\varepsilon)\sin\vartheta \qquad (1.7)$$

Here σ is the angle between the tangent to the shock front and the direction of the incident stream, w_x and w_y are the components of the velocity, w, along the axes $x = -r\cos\theta$ and $y = r\sin\theta$ (Fig. 1); the subscript ∞ indicates quantities in front of the shock wave; $\omega = w_\infty^2 \sin^2\sigma/(1 - w_\infty^2)$.

The velocity components are related in the following way:

$$u = w_y\sin\vartheta - w_x\cos\vartheta, \qquad v = w_x\sin\vartheta + w_y\cos\vartheta \qquad (1.8)$$

From the relation $dy/dx = \tan\ \sigma$ we get

$$\frac{d\varepsilon}{d\vartheta} = -(r_0+\varepsilon)\operatorname{ctg}(\sigma+\vartheta) - \frac{dr_0}{d\vartheta} \qquad (1.9)$$

2. Method of solution. Between the body and the shock wave let us draw the $N-1$ curves

$$r = r_i(\vartheta) = r_0(\vartheta) + \xi_i\varepsilon(\vartheta) \qquad \left(\xi_i = \frac{N-i+1}{N}, \quad i = 2, 3, \ldots, N\right)$$

which break up the region of integration into N strips. We will denote all the quantities on the i'th intermediate curve by the index i, those on the shock front ($i = 1$) by the index 1, and on the body by the index 0.

Let us integrate the first two equations of the system (1.2) along an arbitrary ray θ = const, from the body contour to the boundary of every strip; we then get $2N$ independent integral relations:

$$\frac{d}{d\vartheta}\int_{r_0}^{r_i} S(r,\vartheta)\,dr - \left(S_i\frac{dr_i}{d\vartheta} - S_0\frac{dr_0}{d\vartheta}\right) + r_iH_i - r_0H_0 = \int_{r_0}^{r_i} g(r,\vartheta)\,dr \quad (2.1)$$

$$\frac{d}{d\vartheta}\int_{r_0}^{r_i} t(r,\vartheta)\,dr - \left(t_i\frac{dr_i}{d\vartheta} - t_0\frac{dr_0}{d\vartheta}\right) + r_ih_i - r_0h_0 = 0 \qquad (2.2)$$

We will approximate the functions occurring in the integrands of (2.1) and (2.2) by interpolating polynomials of degree N in the variable r, taking for the points of interpolation the boundaries of the strips

$$f(r,\vartheta) = \sum_{m=0}^{N} a_m(\vartheta)\left[\frac{r - r_0(\vartheta)}{\varepsilon(\vartheta)}\right]^m \qquad (2.3)$$

where the coefficients $a_m(\theta)$ will depend linearly on the values of the corresponding functions on the boundaries of the strips. Writing further the last two equations of the system (1.2) along each intermediate curve $r = r_i(\theta)$ (on the body and on the shock wave, ψ and ϕ are determined from the boundary conditions) and taking into account the equation (1.9), we will get an approximating system, consisting of $4N-1$ equations from the unknown quantities ϵ, σ, v_0, u_i, v_i, ψ_i, ϕ_i, $(i = 2,3, \ldots, N)$.

Let us solve this system for the derivatives of all the unknown functions:

$$\frac{d\varepsilon}{d\vartheta} = -(r_0+\varepsilon)\operatorname{ctg}(\sigma+\vartheta) - \frac{dr_0}{d\vartheta}, \qquad \frac{d\sigma}{d\vartheta} = F$$

$$\frac{dv_0}{d\vartheta} = \frac{E_0}{(\varkappa-1)/(\varkappa+1)-w_0^2}, \qquad \frac{d\psi_i}{d\vartheta} = \rho_i\left[v_i\frac{dr_i}{d\vartheta} - r_i u_i\right] \tag{2.4}$$

$$\frac{du_i}{d\vartheta} = \frac{1}{t_i}\varphi_i^{\frac{1}{\varkappa-1}}\left[f_1 - u_i\varphi_i^{-\frac{1}{\varkappa-1}}f_2 + \frac{S_i}{\varkappa-1}\frac{d\ln\varphi_i}{d\psi_i}\frac{d\psi_i}{d\vartheta}\right]$$

$$\frac{dv_i}{d\vartheta} = \frac{E_i}{(\varkappa-1+2u_i^2)/(\varkappa+1)-w_i^2}, \quad \varphi_i(\psi_i) = \varphi_1(\psi_i) \quad (i = 2, 3, \ldots, N)$$

Here F, E_0, E_i, f_1, and f_2 are known functions of θ and of the quantities to be determined, analytic in the domain of integration, (the form of these functions depends on N), and

$$\frac{d\ln\varphi_i}{d\psi_i} = \left.\frac{d\ln\varphi_1}{d\psi_1}\right|_{\psi_1=\psi_i} = \left.\frac{d\ln\varphi_1}{d\sigma}\frac{d\sigma}{d\vartheta}\frac{d\vartheta}{d\psi_1}\right|_{\psi_1=\psi_i} \tag{2.5}$$

where $d\ln\phi_1/d\sigma$ and $d\psi_1/d\theta$ are computed from (1.6) and (1.7).

All the boundary conditions on the body and on the shock front are satisfied automatically, as can be seen from the manner in which the system was constructed. The integration of the system obtained is done numerically, starting from the axis of symmetry $\theta = 0$, where $v_0 = v_i = \psi_i = 0$, $\sigma = \frac{1}{2}\pi$, $\phi = \phi_1(0)$, and the N initial values of ϵ and u_i are unknown parameters.

In the mixed flow under examination, disturbances in the supersonic region beyond the limiting characteristic do not influence the subsonic region, and consequently we cannot obtain on the upper boundary of the domain of integration the additional conditions needed to determine the missing initial values of the unknown functions. However, from the structure of the approximating system it is evident that in the neighborhood of the sonic line N equations of the system have N moving singular points. In order that a continuous transition across these points be possible, it is necessary that certain conditions be satisfied at these points; namely, at the points where

$$w_0^2 = \frac{\varkappa-1}{\varkappa+1}, \qquad w_i^2 = \frac{\varkappa-1+2u_i^2}{\varkappa+1} \qquad (i = 2, 3, \ldots, N)$$

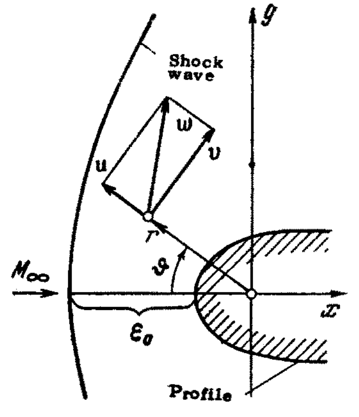

Fig. 1.

$E_0 = 0$ and $E_i = 0$ respectively (we will call the totality of these points a singular line). If the conditions are not satisfied, the derivatives $dv_0/d\theta$, $dv_i/d\theta$, become infinite on the singular line, i.e. the accelerations become infinite, and then the motion cannot be continued across this line; the singular line is then a limit line and the whole solution has no physical meaning. These conditions are similar to the ones obtained by Khristianovich [9] from an exact system of partial differential equations for certain cases of plane irrotational flows. To satisfy the N conditions on the singular line we have at our disposal N parameters when $\theta = 0$. In this way, the requirement that the motion be continuous across the singular line yields the missing conditions for the determination of the problem.

In the approximating system (2.4) the right hand sides are functions, analytic in all their arguments everywhere in the domain under examination, except at the points of the singular line, where N equations have right hand sides of the form 0/0. According to Cauchy's theorem, through every nonsingular point of the domain there passes one, and only one, solution of the system (2.4), which is analytic in θ; consequently, in the neighborhood of such a point this solution can be represented in the form of power series.

3. Construction of the solution in the neighborhood of the singular points. Let us first examine the equation

$$\frac{dW}{dZ} = \frac{P(Z, W)}{Q(Z, W)} \qquad (3.1)$$

where $P(Z,W)$ and $Q(Z,W)$ are analytic functions of the variables Z, W in the neighborhood of the point $Z = W = 0$, with

$$P(O, O) = Q(O, O) = 0$$

$$P(Z, W) = aZ + bW + O(Z^2 + W^2)$$
$$Q(Z, W) = cZ + dW + O(Z^2 + W^2) \qquad (ad - bc \neq 0) \qquad (3.2)$$

Let λ_1, λ_2 be the roots of the equation $\lambda^2 - (b + c)\lambda - (ad - bc) = 0$. We will show that if $\lambda = \lambda_1/\lambda_2$ is a real negative number (which takes place when $(b - c)^2 + 4ad > 0$ and $ad - bc > 0$), then equation (3.1) in the real domain has no other integrals tending to zero together with Z except for the two analytic ones.

Let us investigate the possibility of the existence of a solution which is $O(Z^m)$ at the origin; for this we reduce (3.1) to the canonical form

$$\frac{d\eta}{d\xi} = \frac{P_*(\xi, \eta)}{Q_*(\xi, \eta)} \equiv \frac{\lambda\eta + \Phi_1(\xi, \eta)}{\xi + \Phi_2(\xi, \eta)} \tag{3.3}$$

by means of a nonsingular linear transformation. Here Φ_1 and Φ_2 are functions of order $\xi^2 + \eta^2$, analytic in ξ and η in the neighborhood of the point $\xi = \eta = 0$.

From this, by the way, it follows immediately that in the real domain the solution which is being sought has only two distinct critical directions, along which four integral curves enter the singular point, i.e. the singular point is a saddle point.

If we construct the Puisot diagram for equation (3.3), we will find that at the point $\xi = \eta = 0$ the solutions are understood to be $(\eta^2) = O(\xi^2)$ and $\xi = O(\eta^2)$. Each condition defines a change of variables which transforms (3.3) to a Briot and Bouquet equation of the first reduced type, wit with a real negative coefficient b corresponding to the unknown function [$b = \lambda - 2$ when $\eta = O(\xi^2)$ and $b = (1 - 2\lambda)/\lambda$ when $\xi = O(\eta^2)$]. Briot and Bouquet [10] showed that with the exception of the special case when b is a positive integer, the equation obtained has one, and only one, analytic integral passing through the singular point; furthermore, in the real domain there is no other integral, except for the analytic one, passing through the point, when Re $b < 0$. The convergence of the series representing the solution is proved by introducing a majorant for the right hand side of the Briot and Bouquet equation. In the Z, W variables, the solutions of (3.1) passing through the point $Z = W = 0$ can be written as converging series of the form

$$W_j = \sum_{n=1}^{\infty} A_{nj} Z^n \qquad (j = 1,2) \tag{3.4}$$

The assertion made above has thus been proved.

Let us now return to the approximating system (2.4). Suppose on the contour of the body the singular point is at $\theta = \theta_0$, and on the i'th intermediate curve, at $\theta = \theta_i$. We will assume

$$\vartheta_0 \neq \vartheta_2 \neq \vartheta_3 \neq \cdots \neq \vartheta_N \tag{3.5}$$

which can always be obtained by appropriately choosing the intermediate curves. Then, at the k'th singular point

$$\left(\vartheta_k, \varepsilon^{(k)}, \sigma^{(k)}, \ldots, \varphi_i^{(k)}\right) \qquad (k = 0, 2, 3, \ldots, N;\ i = 2, 3, \ldots, N)$$

only one of the $3N$ differential equations

$$\frac{dv_k}{d\vartheta} = \frac{E_k}{\Delta_k} \tag{3.6}$$

has a right hand side which is indeterminate; however, E_k, Δ_k, and also the right hand sides of the remaining equations will be analytic in all the arguments in the neighborhood of this point.

Let us linearize the remaining system; then equation (3.6) can be written in the form (3.1) ($Z = \theta - \theta_k$, $W = v_k - v_k^{(k)}$), and the conditions for the existence of a real negative λ are satisfied. Consequently the singular point is a saddle point, and the equation (3.6) has no real integrals satisfying the condition $v_k = v_k^{(k)}$ for $\theta = \theta_k$, except for the two analytic ones (3.4).

Then in the neighborhood of the k'th singular point there exist two, and only two, solutions of the system (2.4) with the initial values

$$\vartheta = \vartheta_k, \quad \varepsilon = \varepsilon^{(k)}, \quad \sigma = \sigma^{(k)}, \ldots, \varphi_i = \varphi_i^{(k)}$$

and both solutions are analytic in some domain containing the point $\theta = \theta_k$. Let us connect one of these solutions to the solution before the singular point, obtained in the usual manner (the number of conditions for connecting is equal to the number of arbitrary conditions at the singular point), but since the second solution intersects the first only at the singular point itself, the matching conditions uniquely determine the integral curve through the given singular point. Thus, the approximating system (2.4) has, in the complete domain of integration under examination, a unique solution, which is analytic everywhere and which satisfies the conditions both at $\theta = 0$ and on the singular line.

The solution of the approximating system is carried out numerically with different numbers of intermediate curves: in the first approximation ($N = 1$) there are no intermediate curves, and S, g, t are approximated linearly from their values on the body and on the shock wave. The three unknown functions ϵ, σ, v_0 are found from three differential equations. In the second approximation ($N = 2$) one intermediate curve is introduced, in the third - ($N = 3$) - two curves, etc. Coincidence of the results within the required degree of accuracy in the last two iterations can be used as an indication of convergence in an actual case. The basic difficulties of the computation result because, in the first place, the boundary conditions are given on the singular line, and in the second place, after satisfying at each singular point the matching condition, one must integrate through this point numerically (in solving one must go through all N singular points of the approximating system). Utilizing the series representation of the solution in the neighborhood of both the regular and the singular points, and taking into account the converging character of the integral curves, we succeeded in constructing simple methods of passing through the singular points of the approximating system.

4. Some formulae for the flow behind the shock wave. We will investigate the local properties of the configuration of the streamlines, ψ = const. and the lines of constant speed, w = const.

Let us derive the formula for the angle δ between these lines, and also for the angle of inclination of the line w = const. to the direction of the incident stream (denote this angle by Φ).

The value of these angles enables one to estimate in the following the accuracy of the computation in the most sensitive regions, and yields, without any numerical computation, qualitative information about the character of the domain of influence for different Mach numbers of the incident flow.

First, let us find expressions for these angles on the shock wave. Along a curve w = const. which has the equation $y = y_w(x)$ we have

$$\frac{dy_w}{dx} \equiv \operatorname{tg} \Phi_* = - \frac{w_x \partial w_x / \partial x + w_y \partial w_y / \partial x}{w_x \partial w_x / \partial y + w_y \partial w_y / \partial y} \tag{4.1}$$

To determine the four derivatives $\partial w_x/\partial x$, $\partial w_x/\partial y$, $\partial w_y/\partial x$, and $\partial w_y/\partial y$, we have a system of four equations (the vorticity equation, the transformed equation of continuity, and the two expressions for the total derivative of w_x and w_y along the shock wave):

$$\begin{gathered} \frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} = \Omega \\ \left(1 - \frac{w_x^2}{c^2}\right) \frac{\partial w_x}{\partial x} - \frac{w_x w_y}{c^2} \left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right) + \left(1 - \frac{w_y^2}{c^2}\right) \frac{\partial w_y}{\partial y} = 0 \\ \frac{\partial w_x}{\partial x} + \frac{\partial w_x}{\partial y} \operatorname{tg} \sigma = \frac{dw_x}{dx}, \qquad \frac{\partial w_y}{\partial x} + \frac{\partial w_y}{\partial y} \operatorname{tg} \sigma = \frac{dw_y}{dx} \end{gathered} \tag{4.2}$$

Here Ω is the vorticity, c is the speed of sound:

$$\Omega = \frac{1}{2\varkappa} (1 - w^2)^{\frac{\varkappa}{\varkappa - 1}} \varphi^{-\frac{1}{\varkappa - 1}} \frac{d \ln \varphi}{d\psi}, \qquad c^2 = \frac{\varkappa - 1}{2} (1 - w^2)$$

Let us determine the right hand sides of (4.2). Along the shock wave we have

$$\frac{d \ln \varphi}{d\psi} = \frac{d \ln \varphi}{d\sigma} \frac{d\sigma}{dx} \frac{dx}{d\psi}, \qquad \frac{d\psi}{dx} = \rho_\infty w_\infty \operatorname{tg} \sigma$$

Therefore, taking into account (1.6), we will find

$$\Omega = F_1 \frac{d\sigma}{dx}$$

$$F_1 = \frac{1}{2\varkappa} (1 - w^2)^{\frac{\varkappa}{\varkappa - 1}} \varphi^{-\frac{1}{\varkappa - 1}} \frac{2 [\omega - {}^1/_2 (\varkappa - 1)]^2 \operatorname{ctg}^2 \sigma}{\rho_\infty w_\infty (\omega + 1) [\omega - {}^1/_4 (\varkappa - 1)^2 / \varkappa]} \tag{4.3}$$

where ω and w^2 are obtained from the boundary conditions on the shock wave.

Similarly, from the relations (1.4) and (1.5) we get

$$\frac{dw_x}{dx} = F_2 \frac{d\sigma}{dx}, \qquad F_2 = -2\frac{w_\infty}{\varkappa+1}\sin 2\sigma \tag{4.4}$$

$$\frac{dw_y}{dx} = F_3 \frac{d\sigma}{dx}, \qquad F_3 = 2\frac{w_\infty}{\varkappa+1}\left(\cos 2\sigma + \frac{1}{M_\infty{}^2\sin^2\sigma}\right) \tag{4.5}$$

Evaluating now the unknown derivatives from (4.2), and substituting them into (4.1), we get

$$\operatorname{tg}\Phi_* = -\frac{m(w_x - w_y \operatorname{ctg}\sigma) + (w_\infty - w_x)(F_1 + F_2\operatorname{ctg}\sigma)}{(F_2 - m\operatorname{ctg}\sigma)(w_x - w_y\operatorname{ctg}\sigma) + w_y(F_3 - F_1)} \tag{4.6}$$

Here

$$m = \frac{(c^2 - w_y{}^2)(F_1 + F_2\operatorname{ctg}\sigma - F_3) + w_x(w_\infty - w_x)(F_1 + 2F_2\operatorname{ctg}\sigma)}{c^2(1+\operatorname{ctg}^2\sigma) - (w_x - w_y\operatorname{ctg}\sigma)^2}$$

On the shock wave the angle $\Phi = \Phi_1$ is determined in the following way:

$$\Phi_1 = \begin{cases} \Phi_* & \text{for } \Phi_* < \sigma \\ -(\pi - \Phi_*) & \text{for } \Phi_* > \sigma \end{cases} \tag{4.7}$$

When $\sigma = \frac{1}{2}\pi$ and $\sigma = \alpha = \arcsin M_\infty^{-1}$, the curves $w =$ const. are tangent to the shock wave.

Now it is easy to determine the angle δ_1 between the streamline and the curve $w =$ const.:

$$\delta_1 = \Phi_1 - \beta_1, \qquad \beta_1 = \operatorname{arc\,tg}\frac{w_y}{w_x} \tag{4.8}$$

The angles Φ, δ, and β are positive when measured in the counterclockwise direction.

As we know, σ and β_1 are functions of M_∞ and w on the shock wave; then from the expressions (4.6) - (4.8) it follows that $\Phi_1 = \Phi_1(M_\infty, w)$ and $\delta_1 = \delta_1(M_\infty, w)$, i.e. for the plane case on the shock wave, the angle which the curve $w =$ const. makes with the streamline and with the direction of the incident stream (and consequently, the angle at which the curve $w =$ const. approaches the shock wave) depends only on the Mach number of the incident stream (M_∞) if the speed w is given. The shape of the body thus has no influence on the value of this angle. For axisymmetric flows these angles depend also on the curvature of the shock wave, and therefore on the shape of the body.

Making use of the boundary conditions on the body, we can, in a similar way, obtain the expressions for these angles on the profile $r = r_0(\theta)$:

$$\operatorname{tg}\Phi_0 = -\frac{[r_0 r_0' \operatorname{tg}\vartheta + r_0^2 - r_0 r_0'' + r_0'^2]\, v_0 + (r_0\operatorname{tg}\vartheta - r_0')\, r_0 v_0'}{[r_0 r_0' - (r_0^2 - r_0 r_0'' + r_0'^2)\operatorname{tg}\vartheta]\, v_0 + (r_0 + r_0'\operatorname{tg}\vartheta)\, r_0 v_0'} \tag{4.9}$$

but $\delta_0 = \Phi_0 - \beta_0$, so

$$\operatorname{tg}\delta_0 = \frac{R_0 w_0'}{w_0\sqrt{r_0^2 + r_0'^2}} \tag{4.10}$$

Table 1.

M_∞ / M		1.00	1.25	1.50	1.75	2.00	3.00	4.00	5.00	6.00	7.00	8.00
0.4	Φ°										82°122	77°289
	δ°										76°689	68°386
0.6	Φ°					67°132	42°498	41°081	41°416	41°836	42°166	42°411
	δ°					57°808	16°657	9°591	7°196	6°092	5°484	5°117
0.8	Φ°			19°469	3°148	3°933	19°941	27°831	31°626	33°713	34°969	35°793
	δ°			8°951	— 13°844	— 18°035	— 13°142	— 9°925	— 8°460	— 7°687	— 7°244	— 6°953
1.0	Φ°	—90°000	— 92°908	— 84°221	— 69°598	— 50°901	3°312	19°840	26°306	29°548	31°415	32°601
	δ°	—90°000	— 97°894	— 95°915	— 87°356	— 73°606	— 30°697	— 18°914	— 14°804	— 12°888	— 11°838	— 11°189
1.2	Φ°		—122°285	—114°621	—104°251	— 90°000	— 14°790	+ 13°020	22°231	26°499	28°871	30°341
	δ°		—123°556	—122°807	—118°984	—110°180	— 47°391	— 24°702	— 18°010	— 15°151	— 13°644	— 12°741
1.4	Φ°			—131°477	—123°401	—113°402	— 38°411	5°845	18°444	23°787	26°638	28°356
	δ°			—134°393	—133°388	—129°481	— 68°496	— 29°905	— 20°056	— 16°241	— 14°328	— 13°221
1.6	Φ°				—136°859	—128°916	— 67°962	— 2°813	14°530	21°122	24°472	26°442
	δ°				—141°246	—139°999	— 94°948	— 36°131	— 21°815	— 16°897	— 14°565	— 13°262
2.0	Φ°					—150°000	—116°610	— 32°142	5°107	15°491	20°101	22°655
	δ°					—150°000	—136°500	— 60°053	— 26°513	— 18°142	— 14°745	— 12°980
3.0	Φ°						—160°529	—140°048	— 76°378	— 7°013	7°489	12°994
	δ°						—160°529	—154°060	— 96°614	— 30°404	— 17°735	— 13°394
4.0	Φ°							—165°522	—152°297	—121°762	— 25°616	— 0°120
	δ°							—165°522	—162°289	—136°698	— 43°250	— 19°405

Here w_0 is the speed on the body, and R_0, the radius of curvature of the body profile. The primes indicate differentiation with respect to θ.

For a circular cylinder ($r_0 = 1$) one obtains very simple formulae:

$$\operatorname{tg}\Phi_0 = \frac{v_0 + \operatorname{tg}\vartheta v_0'}{v_0 \operatorname{tg}\vartheta - v_0'}, \quad \operatorname{tg}\delta_0 = \frac{v_0'}{v_0} \tag{4.11}$$

From (4.10) it is evident that $\delta_0 = \frac{1}{2}\pi$ for $R_0 = \infty$, and $0 < \delta_0 < \frac{1}{2}\pi$ for $0 < R_0 < \infty$, $w_0' > 0$, i.e. the curves w = const. approach the body contour at right angles only on the straight line sections of the contour, and at an acute angle at those points of the contour where the stream accelerates and the radius of curvature of the contour is finite.

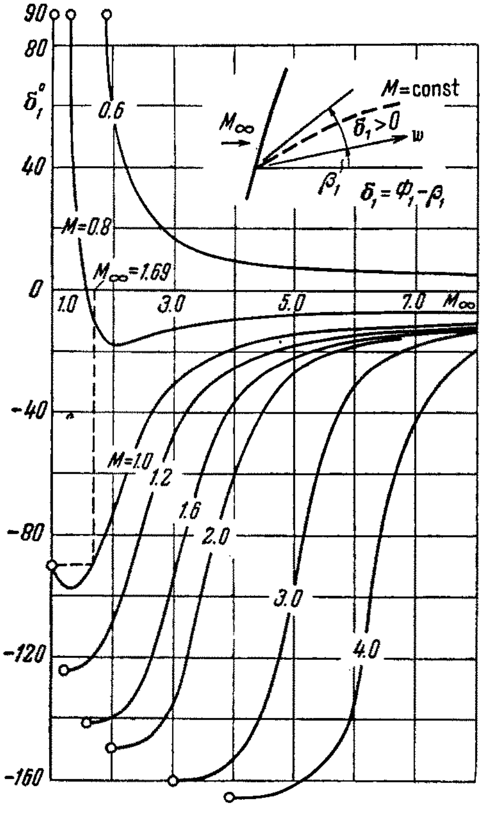

Fig. 2.

The values of the angles Φ_1 and δ_1 on the shock wave, with $\kappa = 1.40$, and for different values of $M = w/c$ and M_∞ are given in Table 1 and Fig. 2. On the sonic line ($M = 1$) we found that $|\delta_1| = \frac{1}{2}\pi$ for $M_\infty = 1.00$ and 1.69 ($\kappa = 1.40$), $|\delta_1| > \frac{1}{2}\pi$ for $1 < M_\infty < 1.69$ and $|\delta_1| < \frac{1}{2}\pi$ for $M_\infty >$ 1.69. The shape of the sonic lines for a circular cylinder are shown in

Fig.3. From this one can immediately establish the character of the domains of influence for different values of M_∞. For $M_\infty \leqslant 1.69$ the minimal domain of influence will be bounded by the characteristic of the first family, which goes from the body to the sonic point of the shock wave; for $M_\infty > 1.69$ there are no characteristics issuing from the sonic point of the shock wave, and the minimal domain of influence will be bounded by the characteristics of the first and second family which emerge from points on the body and on the shock wave for $M > 1$, and are tangent to the sonic line (Fig.4). These results also give some idea of the shape of the domains of influence for other profiles.

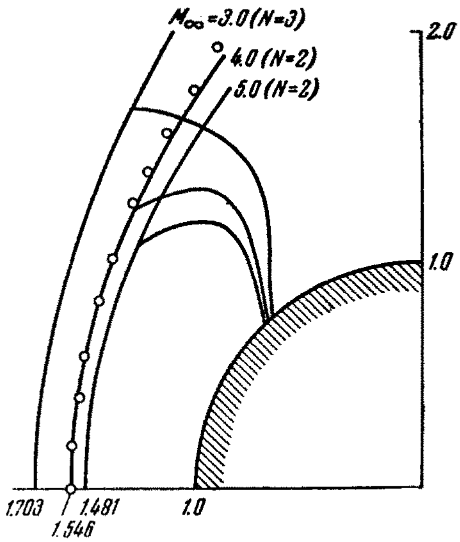

Fig. 3.

5. Computation of the flow around a circular cylinder. As an example of the method described above, computations of the flow past a circular cylinder ($r_0 = 1$) for different values of M_∞ and different degrees of accuracy ($N = 1,2,3$) were made on a high speed electronic computer of the Soviet Academy of Sciences.

Fig.3 shows the shock waves and the sonic lines for the cases $M_\infty = 3.0$ ($N = 3$), 4.0 and 5.0 ($N = 2$); experimental data of Kim [11] for $M_\infty = 4.0$ are also plotted. Incidentally, in his work Kim points out the convex character of the sonic line for $M_\infty = 4.0$; thus our results agree qualitatively with those of Kim. Furthermore, the values of the angle of inclination of the sonic lines at the points on the shock wave and on the body are in good agreement with the exact values, obtained from (4.6) - (4.8) and (4.11).

Fig.4 shows the shock wave, the sonic line, and the boundary characteristics which bound the minimal domain of influence, for the case $M_\infty = 3.0$

in the third approximation ($N = 3$). The picture of the flow obtained from the second approximation is very close to this one.

The manner in which the pressure distribution on the cylinder changes as M_∞ grows from 2.13 to 5.0 is shown in Fig. 5, where the ordinates represent the ratio of the pressure $p_0(\theta)$ to the pressure at the critical point, $p_0(0)$.

Fig.6 and Table 2 illustrate the convergence of the method as the accuracy of the approximation improves ($N = 1,2,3$) for $M_\infty = 3.0$. A comparison with results of experiments made by G.M. Riabinkov is also shown there.

The largest proportion of the work done here and abroad has been devoted to the determination of the distance, ϵ_0, from the body to the shock wave along the axis of symmetry. Fig.7 shows experimental data taken from Kim's article (1), and for comparison the results of our computation (5), and the theoretical results of Tamada (2), Hida (3), and Smurov (4). It is evident from Fig.7 that our results agree quite well with Kim's experimental data (for example, for $M_\infty = 4.0$ Kim gives the value $\epsilon_0 = 0.54$, while we get $\epsilon_0 = 0.546$).

Tables 3 and 4 present the results of the numerical computation for the flow past a circular cylinder with $M_\infty = 3.0$ ($N = 3$, $\kappa = 1.40$).

Table 3 gives the value of the velocity components, u, v, the pressure, p, and the stream function, ψ, at five points of the ray θ = const.; the distance from the cylinder to the shock wave is divided into four equal parts:

$\xi = (r - 1)/\epsilon(\theta)$ = 0(cylinder), 0.25, 0.50, 0.75 and 1.00 (shock wave)

For $\xi = 1.00$, in addition to these quantities, the values of

$$\varepsilon, \qquad \chi = \frac{\pi}{2} - \sigma, \qquad \nu_1 = \varphi^{\frac{1}{\varkappa - 1}}, \qquad \nu_2 = \frac{d \ln \varphi^{1/\varkappa - 1}}{d\psi}$$

are shown.

The dimensionless pressure, temperature (referred to the temperature in front of the shock wave), and entropy, can be found at the same points from the following formulas:

$$\rho = \frac{p}{1 - w^2} \qquad T = 1 - w^2, \qquad s = c_v(\varkappa - 1)\ln \nu_1$$

where $w^2 = u^2 + v^2$, and $\nu_1 = \nu_1(\psi)$; the quantity $\nu_2 = \nu_2(\psi)$ is used in the determination of the vorticity.

Table 4 gives the coordinates of the sonic line and of the boundary characteristics for the same case.

Analysis of the computations and comparison with experimental data show that in the neighborhood of the axis of symmetry the first approximation already gives fair results; the pressure distribution on the body and on the shock wave, and the shape and the position of the shock wave,

Table 2.

ϑ	$N=1$		$N=2$		$N=3$	
	ε	μ_e	ε	μ_e	ε	μ_e
0.0000	0.696	1.000	0.708	1.000	0.703	1.000
0625	698	997	710	997	704	996
1250	701	991	715	989	709	987
1875	708	980	724	974	716	970
2500	717	964	737	952	727	947
0.3125	0.730	0.942	0.752	0.923	0.741	0.918
3750	746	914	770	887	759	883
4375	767	880	792	847	780	844
5000	793	841	818	803	805	801
5625	825	797	847	757	834	754
0.6250	0.862	0.749	0.881	0.708	0.868	0.706
6875	906	697	919	658	907	655
7500	959	642	963	607	952	605
8125	1.020	581	1.012	557	1.002	554
8750	—	—	068	508	057	503
0.9375	—	—	1.130	0.460	1.121	0.455
1.0000	—	—	201	414	193	408
0625	—	—	280	371	273	364
1250	—	—	369	331	365	321

Table 3.

$\xi = 0.00$ ($u \equiv 0$; $\psi \equiv 0$)			$\xi = 0.25$			
$\vartheta\ 10^4$	$v\ 10^3$	$p\ 10^3$	$u\ 10^4$	$v\ 10^3$	$p\ 10^3$	$\psi 10^4$
0	0	328	—706	0	322	0
625	30	327	—703	29	321	17
1250	61	324	—697	57	319	34
1875	92	318	—685	85	314	51
2500	123	311	—667	112	309	67
3125	155	301	—643	140	302	84
3750	186	290	—612	167	294	100
4375	217	277	—574	193	285	116
5000	247	263	—529	219	274	132
5625	277	247	—477	245	264	148
6250	307	231	—417	270	252	164
6875	336	215	—348	294	240	179
7207	352	206	—308	306	234	187
7520	366	198	—268	317	228	194
7832	380	189	—226	329	222	201
8145	394	181	—181	339	216	209
8379	405	175	—146	348	211	214
8750	421	165	— 88	360	204	223
9062	435	157	— 35	370	198	230
9375	448	149	19	380	193	237
9688	462	141	77	390	187	244
1 0000	475	134	137	399	181	251
1 0312	488	126	199	408	176	257
1 0625	501	119	265	417	170	264
1 0938	513	112	334	426	165	271
1 1250	526	105	405	434	159	278
1 1562	538	99	479	442	154	285

TABLE 3.

	$\xi = 0.50$				$\xi = 0.75$			
$\vartheta\ 10^4$	$u\ 10^4$	$v\ 10^3$	$p\ 10^3$	$\psi\ 10^4$	$u\ 10^4$	$v\ 10^3$	$p\ 10^3$	$\psi\ 10^4$
0	—1228	0	311	0	—1662	0	297	0
625	—1224	28	311	33	—1650	29	297	48
1250	—1211	56	309	65	—1629	59	295	97
1875	—1188	84	306	98	—1596	88	293	145
2500	—1155	111	301	130	—1549	118	290	194
3125	—1113	138	296	163	—1490	146	286	242
3750	—1059	164	290	195	—1417	175	281	291
4375	— 996	191	283	227	—1330	202	276	340
5000	— 921	216	275	260	—1230	229	270	389
5625	— 835	241	267	292	—1117	255	263	439
6250	— 738	264	258	324	— 990	280	256	489
6875	— 629	288	249	355	— 850	304	248	538
7207	— 566	299	244	372	— 771	317	244	566
7520	— 503	310	239	388	— 692	328	240	592
7832	— 438	321	235	404	— 611	339	236	618
8145	— 369	331	230	420	— 526	350	232	644
8379	— 316	339	226	432	— 461	358	229	664
8750	— 227	350	221	451	— 354	370	225	695
9062	— 149	360	216	467	— 260	380	220	722
9375	— 068	369	212	484	— 164	389	216	750
9688	16	378	207	500	— 64	398	212	777
1 0000	103	387	202	516	38	407	208	806
1 0312	194	395	198	533	142	415	204	834
1 0625	288	403	193	550	250	424	200	863
1 0938	385	411	189	567	360	431	196	893
1 1250	485	418	184	584	472	439	192	924
1 1562	588	425	180	601	585	446	188	955

$\xi = 1.00$

$\vartheta\ 10^4$	$u\ 10^4$	$v\ 10^3$	$p\ 10^3$	$\psi\ 10^4$	$s\ 10^3$	$\aleph\ 10^3$	$\nu_1\ 10^3$	$\nu_2\ 10^3$
0	—2079	0	281	0	703	0	3046	0
625	—2070	34	280	65	704	35	3041	— 486
1250	—2043	67	279	130	709	69	3027	— 926
1875	—1999	99	278	195	716	102	3004	—1323
2500	—1935	131	276	261	727	135	2975	—1714
3125	—1855	162	273	327	741	167	2937	—2102
3750	—1759	192	269	393	759	200	2893	—2461
4375	—1643	222	266	461	780	232	2843	—2774
5000	—1512	251	261	529	805	263	2787	—3038
5625	—1365	278	257	598	834	294	2727	—3256
6250	—1202	305	252	668	868	324	2664	—3432
6875	—1023	329	246	739	907	354	2598	—3565
7207	— 922	342	243	778	930	369	2562	—3619
7520	— 823	353	241	815	952	384	2528	—3658
7832	— 720	364	238	852	977	398	2593	—3685
8145	— 615	375	235	890	1003	412	2459	—3699
8379	— 533	383	233	919	1023	422	2433	—3699
8750	— 400	394	229	965	1057	439	2391	—3724
9062	— 285	404	226	1004	1088	452	2356	—3719
9375	— 167	413	223	1045	1121	466	2321	—3706
9688	— 47	421	220	1086	1156	479	2286	—3682
1 0000	76	429	217	1127	1193	493	2251	—3650
1 0312	203	437	214	1170	1232	506	2217	—3587
1 0625	331	444	210	1214	1274	518	2183	—3501
1 0938	463	451	207	1258	1318	531	2149	—3408
1 1250	596	457	204	1304	1365	543	2117	—3322
1 1562	732	462	201	1351	1415	555	2084	—3230

are determined with sufficient accuracy in the whole domain of integration by the second approximation, while for the determination of the velocity field, and, consequently, for the construction of the sonic lines with $M_\infty \leqslant 3.0 - 3.5$, one needs at least the third approximation, whereas the second approximation suffices for $M_\infty > 3.5$.

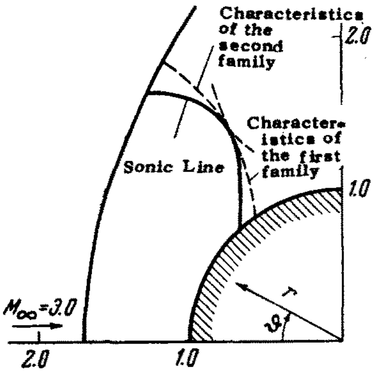

Fig. 4.

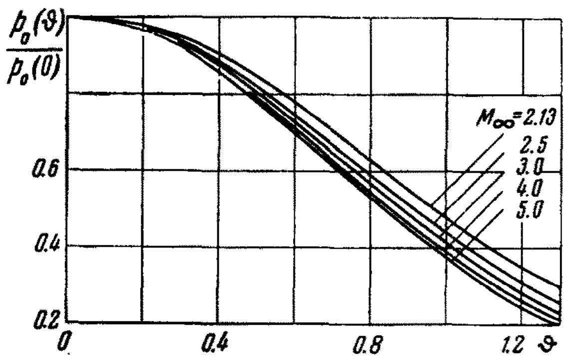

Fig. 5.

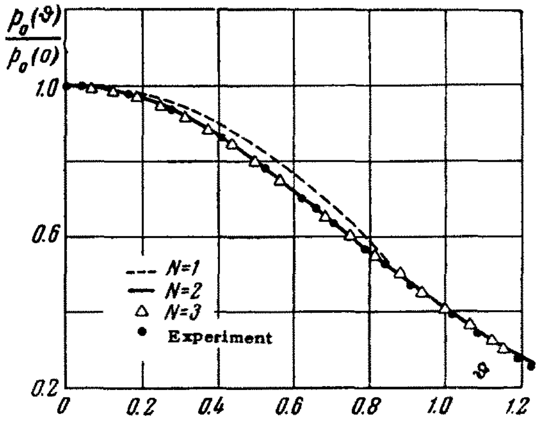

Fig. 6.

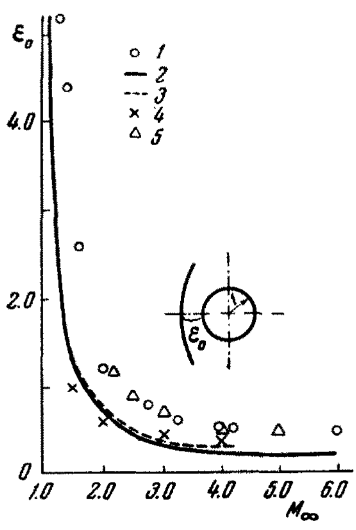

Fig. 7.

The results of the computation enable one to construct, in the domain of influence, a picture of the pressure distribution on and outside of the body, the characteristics, the shock wave, etc.

The problem of the flow around an axisymmetric body, with a detached shock wave, can be solved in a similar way.

TABLE 4.

Sonic Line		Limiting Characteristics			
		First Family		Second Family	
$-x$	y	$-x$	y	$-x$	y
0.664	0.748	0.575	0.818	0.777*	1.433
666	798	576	834	813	494
668	852	584	904	853	544
669	911	593	972	896	592
670	974	605	1.038	941	638
0.671	1.045	0.620	1.101	0.988	1.683
677	135	638	163	1.037	726
692	246	660	222	089	768
720	340	684	280	141	809
777*	443	711	336	186	843
0.841	1.510	0.742	1.391		
940	569	777*	443		
1.029	602				
117	626				
276	668				

Note. Points at which the characteristics are tangent to the sonic line are denoted by an asterisk.

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Translated by M.Y.